Section 5.3

The Indefinite Integral

(1) General Antiderivatives

(2) Antiderivative and the Indefinite Integral

(3) Calculating Indefinite Integrals

Antiderivatives

An **antiderivative** of a function $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x).$$

Calculating an antiderivative involves reversing the derivative process.

Theorem: If $g'(x) = f'(x)$, then $g(x) = f(x) + C$, where C is some constant.

This fact is a consequence of the Mean Value Theorem (see §4.3). In other words,

Theorem: If F and G are both antiderivatives of $f(x)$, then $G(x) = F(x) + C$, where C is some constant.

General Antiderivatives

Theorem: If F and G are both antiderivatives of $f(x)$, then $G(x) = F(x) + C$, where C is some constant.

In other words, a function $f(x)$ *has many different antiderivatives*, all of which differ by a constant.

$F(x) + C$ is called the **general antiderivative** of f.

For example, if an object's acceleration is $a(t)$, then

- the object's velocity is described by the general antiderivative $v(t) + C$,
- the object's displacement is described by $s(t) + Ct + D$, the general antiderivative of $v(t) + C$.

General Antiderivatives

Example 1: A guava moves along the x-axis with velocity $v(t) = 4t + 3$, starting at $t = 0$. What is its x-coordinate as a function of time?

Answer: Not enough information — we don't know where it started!

- The general antiderivative of $v(t) = 4t + 3$ is $s(t) = 2t^2 + 3t + C$. (You can check that $s'(t) = v(t)$.)
- Note that $s(0) = C$. So the constant C represents the starting position.

The best we can do with the information given is

$$s(t) = 2t^2 + 3t + C$$

where C is the x-coordinate at time $t = 0$.

Antiderivative Formulas (1)

Many formulas for antidifferentiation come from reversing the differentiation formulas that we we already know.

Function	x^n (where $n \neq -1$)	x^{-1}	e^x	a^x
Antiderivative	$\dfrac{x^{n+1}}{n+1} + C$	$\ln\lvert x\rvert + C$	$e^x + C$	$\dfrac{a^x}{\ln(a)} + C$

Antiderivative Formulas (2)

Function	$\cos(x)$	$\sin(x)$	$\sec^2(x)$	$\sec(x)\tan(x)$
Antiderivative	$\sin(x)+C$	$-\cos(x)+C$	$\tan(x)+C$	$\sec(x)+C$

Function	$\dfrac{1}{x^2+1}$	$\dfrac{1}{\sqrt{1-x^2}}$
Antiderivative	$\arctan(x)+C$	$\arcsin(x)+C$

However, this leaves many functions that we do not yet know how to antidifferentiate.

The Indefinite Integral

The notation

$$\int f(x)\,dx = F(x) + C$$

means that $F'(x) = f(x)$.

We say that $F(x) + C$ is the general antiderivative or the **indefinite integral** of $f(x)$.

Constant and Sum/Difference Rules

Let $F(x)$ and $G(x)$ be antiderivatives for $f(x)$ and $g(x)$.
Let c be a constant.

- $cF(x)$ is an antiderivative for $cf(x)$.
- $(F \pm G)(x)$ is an antiderivative for $(f \pm g)(x)$.

There do **NOT** exist similar rules for products, quotients, and compositions.

Calculating antiderivatives is significantly more complicated than calculating derivatives!

Though we don't easily obtain antiderivative rules for products and compositions, we will develop techniques in chapters 5 and 6 that will enable us to calculate antiderivatives of certain products and compositions.

General Antiderivatives: Examples

Example 2: Find the general antiderivative of each function.

(i) $f(x) = \frac{1}{2}x^2 - 2x + 6$

Antidifferentiate term by term:

$$\int f(x)\,dx = \frac{1}{6}x^3 - x^2 + 6x + C$$

(ii) $g(x) = (x+5)(2x-6)$

First, expand to get a single polynomial: $g(x) = 2x^2 + 4x - 30$

$$\int g(x)\,dx = \frac{2}{3}x^3 + 2x^2 - 30x + C$$

General Antiderivatives: Examples

Example 2: Find the general antiderivative of each function.

(iii) $h(t) = \dfrac{3+t+t^2}{\sqrt{t}}$

Rewrite as a sum of power functions: $h(t) = 3t^{-1/2} + t^{1/2} + t^{3/2}$

$$\int h(t)\,dt = 6t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$$

(iv) $p(x) = |x|$ Work piecewise:

$$p(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\int p(x)\,dx = \begin{cases} \frac{1}{2}x^2 + C & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{if } x < 0 \end{cases}$$

Example 3: A particle is moving along the x-axis. In each case, find the position function $s(t)$ of the particle.

(a) The velocity function is $v(t) = 3^t$ ft/sec, and $s(2) = 9$.

$$s(t) = \frac{3^t}{\ln(3)} + C \ ft$$

Since $s(2) = 9$, it follows that $C = 9 - \frac{9}{\ln(3)}$.

(b) Acceleration is $a(t) = 2t + 5$, initial velocity is -5 ft/s, and $s(1) = 2$.

$$v(t) = t^2 + 5t + C \qquad\qquad s(t) = \frac{1}{3}t^3 + \frac{5}{2}t^2 + Ct + D$$

Since $v(0) = -5$, it follows that $C = -5$.

Since $s(1) = 2$, it follows that $D = \frac{25}{6}$.

Example 4: A watermelon is dropped off a cliff and hits the ground with a speed of 112 ft/s. What is the height of the cliff? (Use 32 ft/s^2 for the acceleration due to gravity, and assume no wind resistance.)

- The watermelon has constant acceleration $a(t) = -32$ ft/sec^2.
- Velocity is the antiderivative of acceleration:

$$v(t) = -32t + C \text{ ft/sec}$$

  Initial velocity is $v(0) = 0$, so $C = 0$ and $v(t) = -32t$.

- Let T be the time that the watermelon hits the ground. Then $v(T) = -112 = -32T$ so $T = 112/32 = 7/2$.
- Height is the antiderivative of velocity:

$$h(t) = -16t^2 + D \text{ ft}$$

- Therefore, the height of the cliff is

$$h(0) - h(7/2) = D - \left(-16(7/2)^2 + D\right) = 16(7/2)^2 = \boxed{196 \text{ ft.}}$$